Math 135 - More Big-O

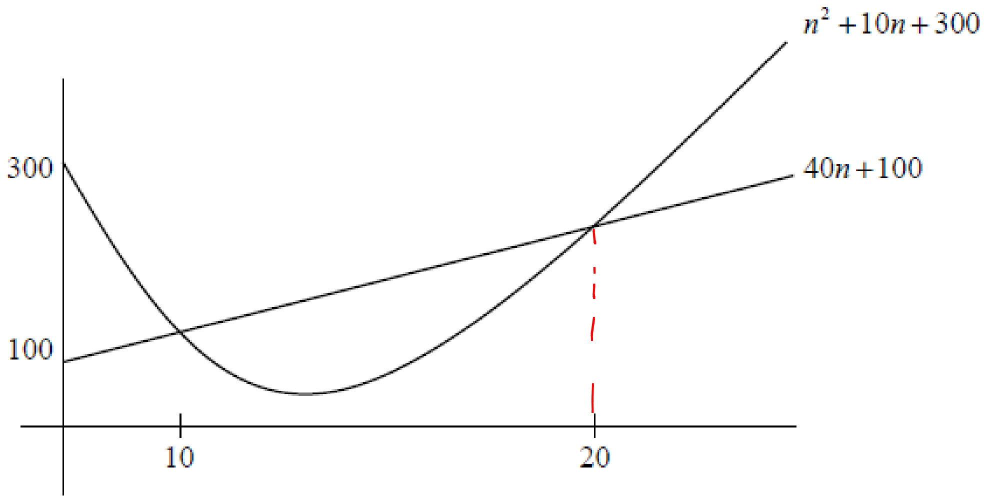
Note Title 10/6/2010

Announcements

- New HW will be posted by Friday
  due after fall break

Growth of functions: Section 3.2

Consider 2 functions:



← this one is "bigger"

Which is bigger?

x is $O(x^2)$

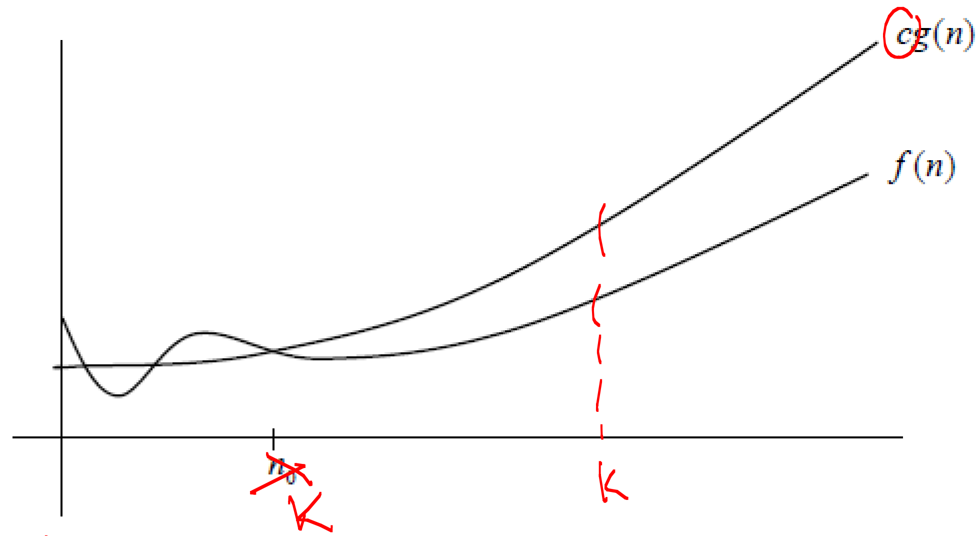
x^2 is not $O(x)$

Big-O:

Dfn: Let f & g be functions from $\mathbb{R} \to \mathbb{R}$ (or $\mathbb{Z} \to \mathbb{R}$). We say that $f(x) = O(g(x))$ if there are constants C and k such that

$$|f(x)| \leq C|g(x)| \quad \text{for } x > k.$$

$a < b$



To prove a function $f(x)$ is $O(g(x))$:

Idea: First select a k that lets you estimate size of $f(x)$ for $\forall x > k$.

Then look for a C that gets desired inequality.

Ex: $\underbrace{7x^2+3x}_{f(x)}$ is $O(\underset{\uparrow g(x)}{x^2})$.

$\underbrace{\text{if } x > 1,}_{k=1}$ $7x^2+3x \le 7x^2+3x^2 = \underbrace{10}_{\text{let } c=10}x^2$

Ex: Show that $\overset{f(x)}{n} = O(\overset{g(x)}{2^n})$

proof: Using induction, we showed $\forall n \geq 1$, $n < 2^n$

$\Downarrow$

Let $k = 1$, $c = 1$

Strategy #2: use induction

Ex: Show that $\log_2 n = O(n)$

Strategy #3: Use "facts" that we know.

If we have an inequality, we can take the log of "both" sides.

$a \le b$ then $\log_2 a \le \log_2 b$

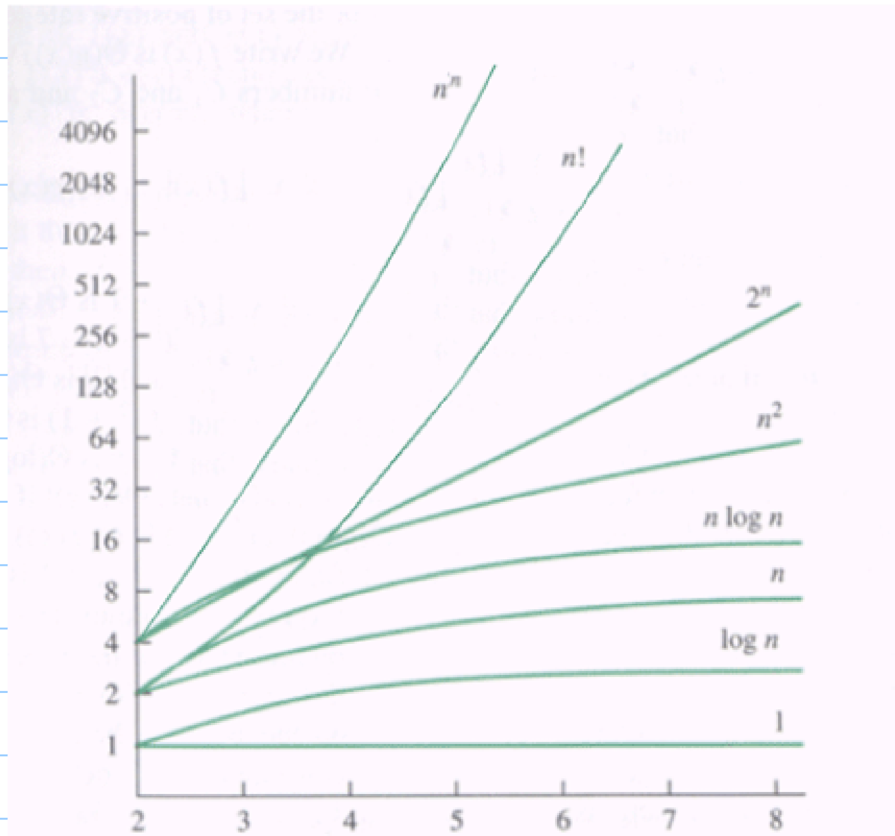
(also $2^a \le 2^b$, $\sqrt{a} \le \sqrt{b}$, ...)

Proof: Know $\forall n \ge 1$, $n < 2^n$

Take $\log_2$: $\log_2 n < \log_2 2^n = n$

So let $k = 1$ & $c = 1$

A big picture:



be a polynomial

Thm: Let $f(x) = \sum_{i=0}^{n} a_i x^i = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$.

Then $f(x) = O(x^n)$.

Pf: Fact: Triangle inequality

$$|a+b| \leq |a| + |b|$$

So let $x > 1$, and $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

$$\leq |a_n| x^n + |a_{n-1}| x^{n-1} + \cdots + |a_1| x + a_0$$

$$\leq |a_n| x^n + |a_{n-1}| x^n + \cdots + |a_1| x^n + a_0 x^n$$

So let $k = 1$ and $c = \sum_{i=0}^{n} |a_i|$ $\blacksquare$

Use this thm:

Give big-O estimates for:

- $f(x) = \frac{1}{8}x^5 + x^3 + 2 = O(x^5)$

- $f(x) = 20000\,x^2 - 10000000\,x = O(x^2)$

- $f(x) = \frac{7x^2}{300} - x + 12 = O(x^2)$

$f(x) = x^2 = O(x^2)$

$h(x) = 3x - 2 = O(x)$ $\Rightarrow$ $f(x) + h(x) = O(x^2)$

Thm: Suppose $f(x) = O(g(x))$ and $h(x) = O(p(x))$.

Then $(f+h)(x) = \underline{O(\max(g(x), p(x)))}$.

Why? $f(x) = O(g(x))$ means:

$\exists\, c_1, k_1$ s.t. $\forall x > k_1$, $f(x) \le c_1 \cdot g(x)$

$h(x) = O(p(x))$:

$\exists\, c_2, k_2$ s.t. $\forall x > k_2$, $h(x) \le c_2 \cdot p(x)$

if $x > k_1$ & $x > k_2$, $f(x) + h(x) \le c_1 g(x) + c_2 p(x)$

Spps $g(x) < p(x)$ & $c_2 > c_1$

$\le 2 c_2 p(x)$

Corollary: Suppose $f_1(x)$ & $f_2(x)$ are $O(g(x))$.
Then $(f_1+f_2)(x) = O(g(x))$.

$$x^2 + 5x^2 = O(x^2)$$

Similarly:

Thm: Suppose $f(x) = O(g(x))$ & $h(x) = O(p(x))$.
Then $(f \cdot h)(x) = O(g(x)\,p(x))$

$f(x) = 3x^2 - 5 = O(x^2)$

$h(x) = x \log x = O(x \log x)$

$f(x) \cdot h(x) = O(x^2 \cdot x \log x) = O(x^3 \log x)$

Ex: Give a big-O estimate for

$$f(x) = \underbrace{3n \log_2(n!)} + (n^2+3)\log_2 n = O(n^2 \log n)$$

$$3n = O(n)$$

$$\log_2(n!) \le \log_2(n^n) = n \log_2 n$$

So $3n \log(n!) = O(n^2 \log_2 n)$

$$\underbrace{(n^2+3)}_{O(n^2)} \underbrace{\log n}_{O(\log_2 n)} = O(n^2 \log_2 n)$$

Big-Omega

Dfn: Let f & g be functions from $\mathbb{R} \to \mathbb{R}$ (or $\mathbb{Z} \to \mathbb{R}$)

We say $f(x)$ is $\Omega(g(x))$ if $\exists$ positive constants C, k such that

$$|f(x)| \geq C \cdot |g(x)| \quad \text{when } x > k.$$

(Read – f is big-Omega of g).

Ex: Show $f(x) = 8x^3 + 5x^2 + 7$ is $\Omega(x^3)$.

Proof: Let $x \geq 1$.

$$8x^3 + 5x^2 + 7 \geq 8x^3$$

So let $k = 1$ and $c = 8$

Note: Similar thm:

$$f(x) = a_n x^n + \cdots + a_0 = \Omega(x^n)$$

need $a_n > 0$