Math 135 - Big-O Notation

Note Title 9/28/2010

Announcements

- Boeing talk in math/cs club
  + Boeing scholarship (due tomorrow)
- Worksheet 4 + 5 are entered -
  pick them up on side of the room
- Review is on Friday
  Test is next Monday
- HW due Friday

Infinite Sets (Ch. 2.4, end of section)

Dfn: Two sets have the same cardinality if and only if there is a bijection from A to B.

Ex: $\mathbb{N}$, $\mathbb{Z}$, & $\mathbb{Q}$ all have same cardinality.

What is bigger than $\mathbb{N}$?

Dfn: A set A is countable if there is a bijection $f: \mathbb{N} \to A$ (or if A is finite).

Thm: $\mathbb{R}$ is not countable. $(0,1)$

Another example: $P(\mathbb{N})$

Assumed there was a bijection, used this "list" to show it couldn't be onto.

But why do we care??

Well, we care about <u>computable</u> things.

How many computer programs are there?

characters

0's + 1's

So a computer program can be "just" a number!

How many functions from $\mathbb{N} \to \{0, 1\}$ are there?

x	0	1	2	3	4	...
$f(x)$	.0	1	0	0	...	

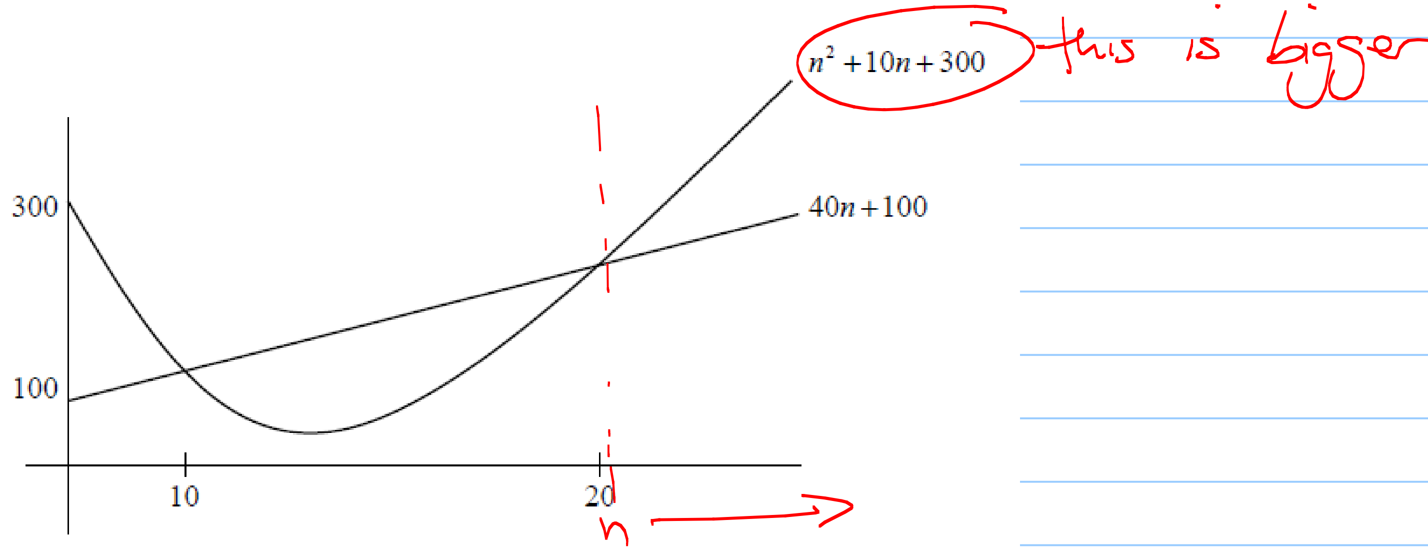
These look like fractions b/t 0 and 1

$\Rightarrow$ there are a lot of uncomputable functions!

Ch. 3.2

Growth of functions:

Consider 2 functions:

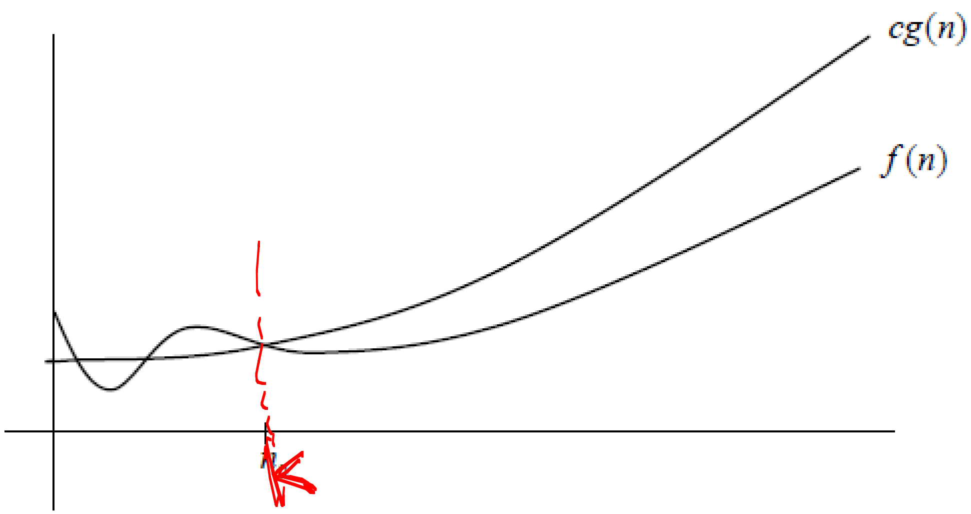


Which is bigger?

Big-O:

Dfn: Let f & g be functions from $\mathbb{R} \longrightarrow \mathbb{R}$ (or $\mathbb{Z} \to \mathbb{R}$). We say that $f(x) = O(g(x))$ if there are constants C and k such that

$$|f(x)| \leq C|g(x)| \quad \text{for } x > k.$$



(here, $n_0 = k$)

Ex: $f(x) = x^2 + 2x + 1$ is $O(x^2)$

$$\underbrace{x^2+2x+1}_{} \text{ is } \underbrace{O(\overset{g(x)}{\overset{\shortparallel}{x^2}})}_{}$$

proof: Need to find k & C.

$\forall x > k, \quad x^2 + 2x + 1 \le C \cdot x^2$

If $x > 1$, then

$x^2 + 2x + 1 \le x^2 + 2x^2 + x^2 = 4x^2$

So let $k = 1$, $c = 4$.

$\boxtimes$

Idea: First select a k that lets you estimate size of $f(x)$ for $x > k$.

Then look for a C that gets desired inequality.

So can also get: $f(x) = O(x^3)$

$f(x) = x^2 + 2x + 1$ is $O(x^3)$

We had: $\forall x > 1$, $x^2 + 2x + 1 \leq 4x^2 \leq 4x^3$

So let $k = 1$, $C = 4$.

Sometimes write $\overbrace{f(x) = O(g(x))}$

Not an equality!

- $x^2 + 2x + 1 = O(x^2)$

- $x^2 + 2x + 1 = O(x^3)$

(Really mean $f(x) \in \{$ functions that are $O(g(x))\}$)

Ex: Show that $7x^2 = O(x^3)$

pf: if $x > 1$,

$7x^2 < 7x^3$

So let $k = 1$, & $c = 7$.

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pf: if $x > 7$,

then $7 \cdot x^2 < x \cdot x^2 = x^3$

So let $k = 7$, $c = 1$

Ex: Show that n^2 is not $O(n)$.

pf: Harder, since we need to show no constants C & k can exist with $n^2 \le C \cdot n$ for some $n > k$.

$$\neg \left(\exists c \exists k \text{ s.t. } \forall n > k,\ n^2 \le c \cdot n \right)$$

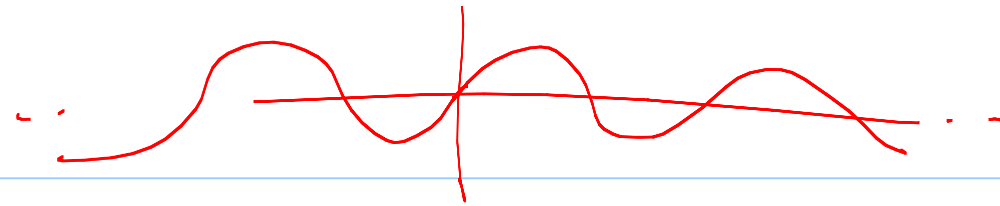
$$\forall c \forall k, \quad \exists n > k \text{ s.t. } n^2 > cn$$

Take any c & any k.

Pick $n = \max\{k+1, c+1\}$

then $n \cdot n > c \cdot n$

(since $n > c$)



<u>Ex</u>: $f(x) = \sin x$ is $O(1)$.

Need to find k & c s.t. $\sin x \le c \cdot 1$.

Let $k=0$ and let $c=2$.

Since $\sin x$ is always ≤ 1,

$\sin x < 2 \cdot 1 \quad \forall x > 0$.

Ex Consider $\sum_{i=1}^{n} i = 1+2+\cdots+n$.

What is it if we want big-O?

(Two ways to do this:)

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2} = O(n^2)$$

$f(n) = \frac{n^2}{2} + \frac{n}{2}$ find c & k:

Let $n > 1$, so $\frac{n^2}{2} + \frac{n}{2} < \frac{n^2}{2} + \frac{n^2}{2} = n^2$

Let $k = 1$, $c = 1$

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + (n-1) + n$$

$$\leq \underbrace{n + n + \cdots + n + n}_{n \text{ times}}$$

$$= n^2$$

Let $c = 1$, $k = 1$.

Ex: Give a big-O bound for $n! = n(n-1)\cdots 1$.

$n! = n(n-1)\cdots 2\cdot 1$

$\leq \underbrace{n\cdot n\cdot n\cdots n}_{n \text{ times}} = n^n$

$n! = O(n^n)$ — let $k=1$, $c=1$

Ex: What about $\log_2(n!)$?

$$\log_2(n!) = \underbrace{\log_2(n\cdot(n-1)\cdots 2\cdot 1)}$$

$$\leq \log_2(n^n)$$

$$= n\log_2 n$$

$$\Rightarrow \log_2(n!) = O(n\log_2 n)$$

$$\log_2(n(n-1)\cdots 2\cdot 1) = \log_2 n + \log_3(n-1) + \log(n-2) \cdots -$$

In induction, we showed $n \le 2^n$ for $n \ge 1$.

What big-Oh does this imply? k

$n = O(2^n)$

Ex: Use above to show $\log_2 n = O(n)$.

know $n \le 2^n$

take $\log_2$ of both sides: $\log_2 n \le \log_2(2^n)$

$\log_2(2^n) = n \log_2 2 = n$

A big picture: