Math 135 – Counting

Note Title 3/26/2010

Announcements

- Cool Math/CS club today at 4 in lobby of Ritter
- HW due Monday
- Review Monday, exam Wednesday
- No office hours after exam

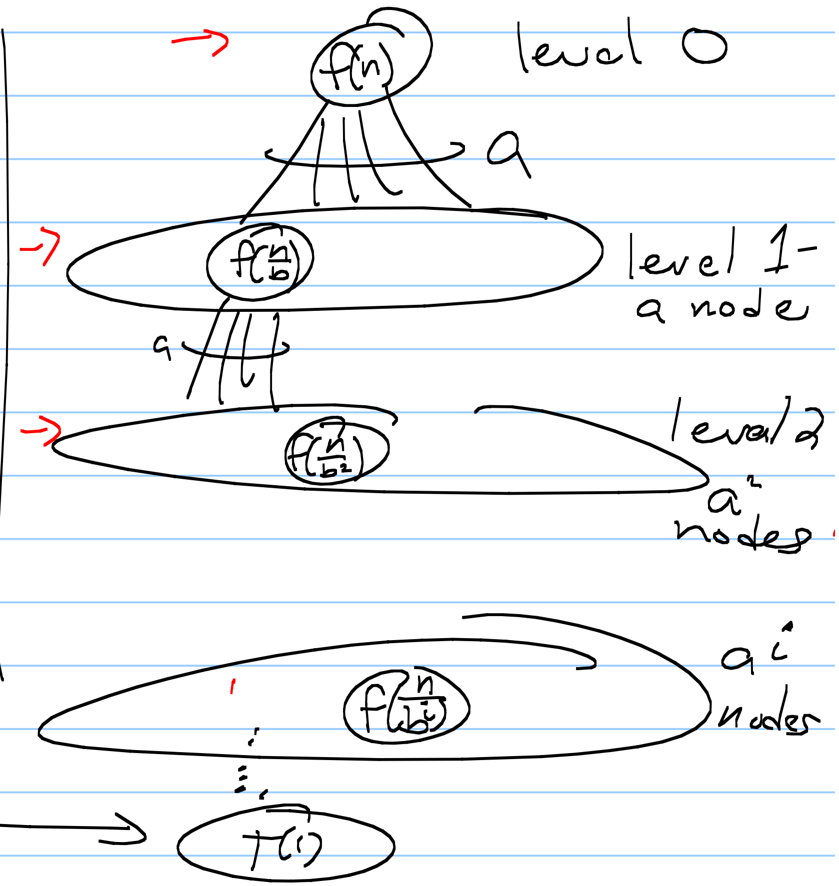
Final word on recurrences

Recursion tree:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$T\left(\frac{n}{b}\right) = a\underbrace{T\left(\frac{n}{b^2}\right)} + f\left(\frac{n}{b}\right)$$

$$T\left(\frac{n}{b^2}\right) = aT\left(\frac{n}{b^3}\right) + f\left(\frac{n}{b^2}\right)$$



$$\frac{n}{b^d} = 1$$

$$\Rightarrow d = \log_b n$$

depth d

\# nodes

value in each node

$$T(n) = \sum_{i=0}^{\log_b n} a^i f\left(\frac{n}{b^i}\right) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + \dots + a^{\log_b n} f(1)$$

Master thm just recognizes that this is increasing or ~~dec~~ decreasing geometric series.

Case 1: Increasing $\rightarrow$ fraction is > 1

~~Case 2~~: when amount on each level is same

Case 3: Decreasing $\rightarrow$ fraction is < 1

Ex: If $f(n) = \boxed{n^k}$, have

$$T(n) = \sum_{i=0}^{\log_b n} a^i \left(\frac{n}{b^i}\right)^k = n^k \sum_{i=0}^{\log_b n} \frac{a^i}{(b^i)^k} = n^k \sum \left(\frac{a}{b^k}\right)^i$$

If $a < b^k$: $\frac{a}{b^k} < 1$ so geometric series

$$T(n) = n^k \sum_{i=0}^{\log_b n} \left(\frac{a}{b^k}\right)^i \le n^k \underbrace{\sum_{i=0}^{\infty} \left(\frac{a}{b^k}\right)^i}$$

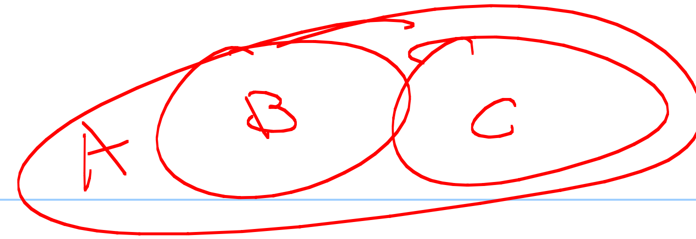
$$= n^k \underbrace{\left(\frac{1}{1 - \frac{a}{b^k}}\right)}_{\text{Constant}} = O(n^k)$$

Counting : Chapter 5

2 Basic Principles

1. Rule of Sum
2. Rule of Product

Rule of Sum



If B and C are disjoint and $A = B \cup C$, then $|A| = |B| + |C|$.

We can split A into non-overlapping subsets, so we can sum the sizes of B & C.

Ex: Need a math representative to a committee. There are 37 students available & 12 faculty.

Total possible choices $= 37 + 12 = 49$

Ex: $A = \{(x,y) \in \{1,2,\dots,n\}^2 : x = 4 \text{ or } x = 5\}$

(Recall: $\{1,2,\dots,n\}^2$ is set of ordered pairs (x,y) where $1 \le x \le n$ and $1 \le y \le n$.)

Here: $A = \{(4,y) \mid y \in \{1,\dots,n\}\} \cup \{(5,y) \mid y \in \{1,\dots,n\}\}$

$|A| = | \qquad | + | \qquad |$

$= n + n = \boxed{2n}$

② Rule of product:

Suppose a set can be formulated as a sequence of k choices.
Then, if there are n_1 ways to make first choice, n_2 to make second, etc.,

$$\underbrace{|A| = n_1 \cdot n_2 \cdots n_k}$$

Ex: How many binary strings of length n?

2^n

n bits to fill in:

$$\underset{\uparrow}{\underset{2}{\_}}\ \underset{\uparrow}{\underset{2}{\_}}\ \underset{\uparrow}{\underset{2}{\_}}\ \_\ \_\ \_\ \_\ \_\ \cdots\ \underset{\uparrow}{\underset{2}{\_}}$$

$$\underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n \text{ times}} = 2^n$$

Ex: Chairs in an auditorium will be labeled with a letter and a positive integer ≤ 100.

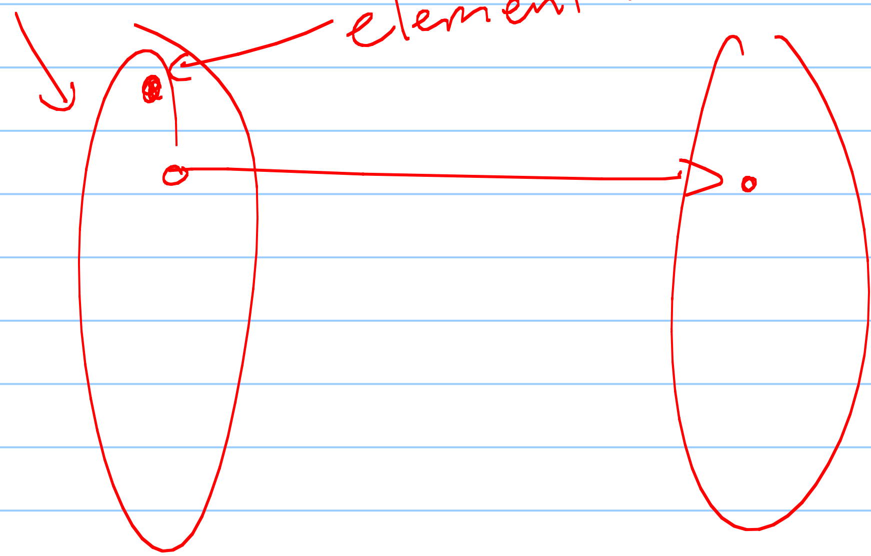
How many chairs are possible?

$\underset{\text{letter}}{\_\_\_\_} \quad \underset{\text{number}}{\_\_\_\_}$

$26 \cdot 100 = \boxed{2600}$

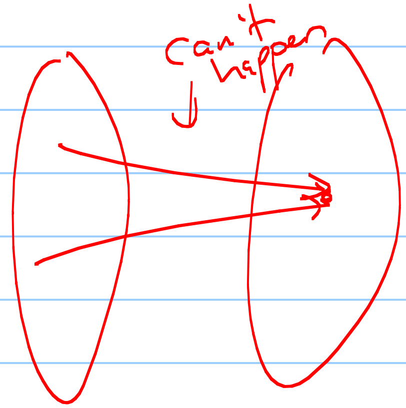
Ex: How many different functions from a set with m elements to a set with n elements?

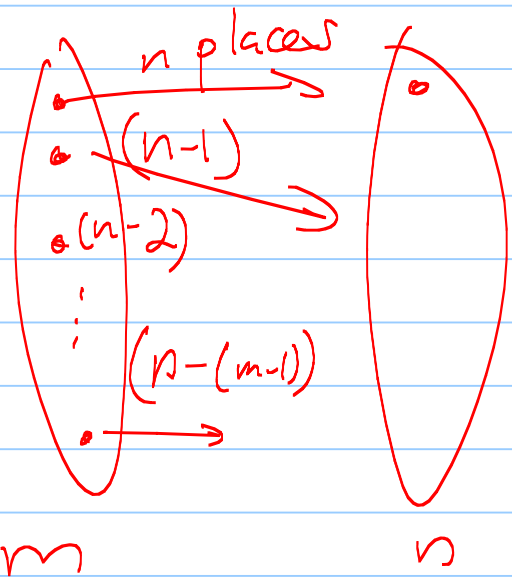
element in domain has n choices



$$\overbrace{\underset{\substack{\uparrow\\ \text{first element in domain}}}{n} \times \underset{\substack{\uparrow\\ 2^{nd}}}{n} \times \underset{\substack{\uparrow\\ 3^{rd}}}{n} \times \cdots \times n}^{m} = \boxed{n^m}$$

Ex: How many one-to-one functions from a set w/ m elements to a set w/ n elements?





$$\text{total} = n(n-1)(n-2)\cdots(n-m+1)$$
$$= n!/m!$$

More complicated

In one version of the programming language BASIC, variables could be 1 or 2 alphanumeric characters.

- Had to begin with a letter.
- 5 reserved keywords were forbidden
- No distinguishing lower + upper case.

How many variables were possible?

$$\underbrace{26}_{\text{1 char passwords}} + \underbrace{\underset{\text{letter}}{26} \cdot \underset{\text{letter or \#}}{36}}_{\text{2 character passwords}} - 5$$

Ex: Suppose you need to make a password.
- 6 to 8 characters long
- Uppercase letters or numbers
- At least 1 digit. ← number

How many are possible?

$$\boxed{\underbrace{36^6 - 26^6}_{6 \text{ chars}}} + \underbrace{36^7 - 26^7}_{7 \text{ chars}} + \underbrace{36^8 - 26^8}_{8 \text{ chars}}$$

$36^5 \cdot 10$

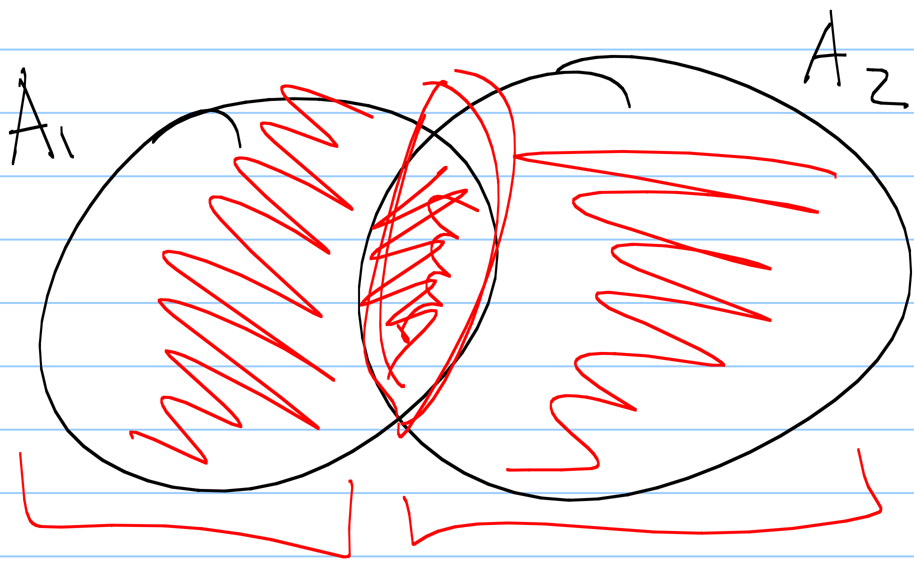
$\underset{10}{\uparrow}\ \underline{36}\ \underline{36}\ \underline{36}\ \underline{36}\ \underline{36}$ + $\underline{36}\ \underset{10}{\uparrow}\ \underline{36}\ \underline{36}\ \underline{36}\ \underline{36}$

$\boxed{\cancel{6\,(36^5 \cdot 10)}}$

Principle of Inclusion / Exclusion

generalize the rule of sum

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$



$$2^{n-1} + 2^{n-2} - 2^{n-3}$$

Ex: How many bit strings of length n either start with a 1 or end with 00?

A_1 = bitstrings that start with 1

$$|A_1| = 2^{n-1}$$

$$1 \;\underset{2}{\_}\;\underset{2}{\_}\;\_\;\_\;\cdots\;\_\;\underset{2}{\_}$$

A_2 = bitstrings ending in 00

$$|A_2| = 2^{n-2}$$

$$\underbrace{\_\;\_\;\_\;\_\;\cdots\;\_}_{n-2}\;00$$

$$|A_1 \cap A_2| = 2^{n-3}$$

$$1\;\underbrace{\_\;\_\;\_\;\cdots\;\_}_{n-3}\;00$$